Single superfield representation for mixed retarded and advanced correlators in disordered systems

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Abstract
We propose a new single superfield representation for mixed retarded and
advanced correlators for noninteracting disordered systems. The method is
tested in the simpler context of random matrix theory, by comparing with
well-known universal behaviour for level spacing correlations. Our method is
general and could be especially interesting for studying localization properties
encoded in the mixed correlators of quantum Hall systems.

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1. Introduction

The physics of electronic localization is one of the most interesting issues in the context
of disordered quantum systems [1]. The localization/delocalization transition has been
intensively studied by following the field theoretical approach of Wegner [2], and the more
formal developments by McKane and Stone [3]. Universal aspects of this transition and related
topics like transport phenomena in disordered media can also be studied in the framework of
random matrix theory (for a review see, for instance, [4]).

In an integer quantum Hall system, electron localization plays a fundamental role [5],
namely, the effect of disorder is that of localizing the states of energy E away from the
Landau energy $E_n = (n + 1/2)\hbar\omega_c$. Near this value, the localization length $\xi(E)$ diverges
like $\xi(E) \propto (E - E_n)^{-\nu}$. Although this behaviour is consistent with experiments [6] and
with several numerical [7] and renormalization group calculations [8], to the best of our
knowledge, there is no simple analytical method to evaluate this scaling [9] nor it is clear why,
in the thermodynamic limit, just one state for each Landau band remains extended.

The electronic properties are encoded in the system's correlators; for instance, the density of states (localized or not) can be obtained from the retarded Green's function,

$$\rho(E) = \frac{1}{\pi} \lim_{\eta \to 0^+} \text{Im} \langle G(x, x; E - i\eta) \rangle, \tag{1.1}$$

where $\langle \cdots \rangle$ denotes the ensemble average over the random variable. On the other hand, the analytical structure of the density of *localized* states $\rho_\ell(E)$ is more complex since it involves the mixed product of retarded and advanced Green's functions,

$$\rho_\ell(E) = \frac{1}{\pi} \int \mathrm{d}x \lim_{\eta \to 0^+} \eta \langle G(x, y; E - i\eta) G(x, y; E + i\eta) \rangle. \tag{1.2}$$

In fact, any quantity encoding some information about localization or transport properties is related with this mixing. For this reason, we concentrate our efforts on studying the representation of the ensemble average for these important quantities.

There are essentially two methods to calculate averages: the replica [10] and the supersymmetric method [11]. The first one is based on the introduction of a number of copies to represent each type of Green's function. In order to compute the average density of states one set of n field copies is needed. In this case, a generalized n component Landau–Ginsburg model is obtained [12], where the limit $n \to 0$ must be implemented. For the density of localized states two sets of copies are used, containing n_+ and n_- field copies, respectively. The disorder average, after integrating out all massive modes, leads to a generalized nonlinear sigma model with symmetry group $O(n_+, n_-)$ [2, 3], describing the critical properties near the mobility edge.

In the supersymmetric method, the Green's function denominator is exponentiated by means of Grassmann variables, thus avoiding dangerous continuations in the number of copies. In order to compute the density of states a superfield is needed, this also applies to other quantities that can be computed from correlators just involving a retarded prescription. This method has been used by Brézin, Gross and Itzykson [15] to study the density of states in Hall systems. They showed that the reason behind the exact expression previously calculated by Wegner [2] is an exact boson–fermion symmetry in the Green's function representation, when the projection to the first Landau level is considered. However, the problem is not so simple when one tries to extend these ideas to understand localization. As occurs in the replica method, a doubling is used to represent the density of localized states, that is, a superfield is introduced for each prescription in equation (1.2). As a consequence, the boson–fermion symmetry is lost, thus precluding the possibility of obtaining exact results (see the discussion in [15]).

Therefore, it is interesting to look for alternative representations for mixed correlators, thus opening the possibility to advance the understanding of disordered systems.

The aim of this paper is to propose a new representation for mixed correlators based on the introduction of essentially just one superfield. Our method is general and can be applied to any noninteracting disordered system. It is based on the introduction of a single operator, quadratic in the system's Hamiltonian, that permits us to encode the nontrivial information about retarded–advanced mixing by using a single prescription. Then, the basis of the method is in fact quite general and could be useful outside the supersymmetric framework.

Our method will be tested in the simpler context of random matrix theory where the supersymmetric method has been introduced by the pioneering work of Efetov [13]. We will also show how the boson–fermion symmetry can be restored when considering mixed correlators in Hall systems.

In section 2, we outline the general single superfield method for mixed correlators. In sections 3 and 4 we compute the corresponding averages for the Gaussian unitary ensemble, while in section 5 we study level spacing correlators and compare with well known universal

asymptotic behaviour. In section 6 we briefly discuss quantum Hall (QH) systems. Finally, section 7 is devoted to presenting our conclusions and discussing possible perspectives.

2. Single superfield/supervector representation

We start by considering the operator,

$$O(\mathcal{E}) = H - \mathcal{E}, \tag{2.1}$$

where H is the system's Hamiltonian. We will use the notation,

$$O_{1+} = O(E_1 + i\eta) \qquad \text{and} \qquad O_{2-} = O(E_2 - i\eta), \tag{2.2}$$

where $\eta > 0$ and E_1, E_2 are two real energies. Now, we will define the operator O_m, which is quadratic in H and mixes the retarded and advanced prescriptions,

$$O_m = O_{1+}O_{2-} = O_{2-}O_{1+} \tag{2.3}$$

$$= (H - E)^2 - (r/2 + i\eta)^2, \tag{2.4}$$

where $E = (E_1 + E_2)/2$ and $r = E_1 - E_2$. As the factors in O_m commute, it is easy to verify,

$$O_m\left[O_{1+}^{-1} - O_{2-}^{-1}\right] = O_{2-} - O_{1+} = (r + 2i\eta), \tag{2.5}$$

therefore, the inverse of the operator O_m is

$$O_m^{-1} = \frac{1}{r + 2i\eta}\left[O_{1+}^{-1} - O_{2-}^{-1}\right] \tag{2.6}$$

(for $\eta \neq 0$, these inverses are well defined). In other words, the three types of associated Green's functions $G(x, y) = \langle x|O^{-1}|y\rangle$ are related by

$$G_m(x, y; E, r) = \frac{1}{r + 2i\eta}[G(x, y; E_1 - i\eta) - G(x, y; E_2 + i\eta)]. \tag{2.7}$$

Our strategy is based on first using a single superfield to compute four-point correlators associated with the O_m operator. Then, from Wick's theorem and equation (2.7), we can relate this correlator to the sum of three terms involving only retarded, only advanced or mixed products of two-point Green's functions, respectively. The first two terms are in general simpler to study and well known in many different disordered systems. Then, we see that the four-point correlators for O_m operators encode important information about mixed products. Moreover, as the retarded–retarded and advanced–advanced terms also admit a single superfield representation, the ensemble average for the mixed prescriptions could be performed on independent single superfield representations.

In particular, in the random matrix case, this procedure will be tested by calculating finite N correlation functions and comparing with well-known large N universal results for the Gaussian unitary ensemble (GUE), given by the probability distribution,

$$P(H)\,dH = \mathcal{N}\prod_{i=1}^{N} dH_{ii} \prod_{i<j} dH_{ij}\,d\overline{H}_{ij} \exp\left(-\frac{1}{2}\gamma N \operatorname{tr}(H^2)\right), \tag{2.8}$$

where H is an $N \times N$ Hermitian matrix.

Taking traces in equation (2.6) we get,

$$\operatorname{tr}\left(O_m^{-1}\right) = \left[\operatorname{tr}\left(O_{1+}^{-1}\right) - \operatorname{tr}\left(O_{2-}^{-1}\right)\right]/(r + 2i\eta). \tag{2.9}$$

Now we can multiply equation (2.6) twice, take the trace, multiply equation (2.9) twice and then sum together the two expressions to obtain,

$$\operatorname{tr}\left(O_{1+}^{-1}\right)\operatorname{tr}\left(O_{2-}^{-1}\right) + \operatorname{tr}\left(O_{1+}^{-1}O_{2-}^{-1}\right) = \frac{1}{2}\left[\operatorname{tr}\left(O_{1+}^{-1}\right)\operatorname{tr}\left(O_{1+}^{-1}\right) + \operatorname{tr}\left(O_{1+}^{-1}O_{1+}^{-1}\right) + \operatorname{tr}\left(O_{2-}^{-1}\right)\operatorname{tr}\left(O_{2-}^{-1}\right)\right.$$

$$\left. + \operatorname{tr}\left(O_{2-}^{-1}O_{2-}^{-1}\right)\right] - (r + 2i\eta)^2/2\left[\operatorname{tr}\left(O_m^{-1}\right)\operatorname{tr}\left(O_m^{-1}\right) + \operatorname{tr}\left(O_m^{-1}O_m^{-1}\right)\right]. \tag{2.10}$$

When η goes to zero, the first member in equation (2.10) is a mixed retarded and advanced four-point correlator at energies E_1 and E_2, respectively.

On the other hand, in the second member, we have the sum of three four-point correlators for the operators O_{1+}, O_{2-} and O_m, respectively. For each correlator we can introduce a single supervector representation of the type,

$$\text{tr}(O^{-1})\,\text{tr}(O^{-1}) + \text{tr}(O^{-1}O^{-1}) = (\lambda/2)^2(-1)^{N(N+1)/2}\pi^{-N}\int d\varphi\,d\overline{\varphi}\,d\psi\,d\overline{\psi}(\varphi\cdot\overline{\varphi})^2\exp(-S),$$
(2.11)

$$S = \frac{\lambda}{2}(\overline{\varphi}O\varphi + \overline{\psi}O\psi),$$
(2.12)

where φ (ψ) is a bosonic (fermionic) N-component complex vector. The parameter λ must be chosen for the integrals to be well defined.

Then, the idea is using three independent single supervector representations of the type given in equation (2.11) to compute the average for the mixed correlator. This is in contrast with the usual supervector doubling to represent a typical term in the first member of equation (2.10),

$$O_{1+}^{-1}\big|_{ij}\,O_{2-}^{-1}\big|_{kl} = (1/2)^2\pi^{-2N}\int d\varphi_+\,d\overline{\varphi}_+\,d\psi_+\,d\overline{\psi}_+\,d\varphi_-\,d\overline{\varphi}_-\,d\psi_-\,d\overline{\psi}_-\,\varphi_{+i}\overline{\varphi}_{+j}\,\varphi_{-k}\overline{\varphi}_{-l}$$

$$\times\exp\left(-\frac{i}{2}(\overline{\varphi}_+O_{1+}\varphi_+ + \overline{\psi}_+O_{1+}\psi_+) + \frac{i}{2}(\overline{\varphi}_-O_{2-}\varphi_- + \overline{\psi}_-O_{2-}\psi_-)\right).$$
(2.13)

3. Averages for the four-point correlators of O_m operators

When averaging over the unitary ensemble in the second member of equation (2.10), the more involved part of the calculation comes from the O_m correlator, as it is associated with an exponent which depends quadratically on H. Then, in this section we will concentrate on this part, providing a detailed derivation of the correlator,

$$\left\langle\text{tr}\left(O_m^{-1}\right)\text{tr}\left(O_m^{-1}\right) + \text{tr}\left(O_m^{-1}O_m^{-1}\right)\right\rangle = (\lambda/2)^2(-1)^{N(N+1)/2}\pi^{-N}$$

$$\times\int d\varphi\,d\overline{\varphi}\,d\psi\,d\overline{\psi}(\varphi\cdot\overline{\varphi})^2\langle\exp(-S_m)\rangle,$$
(3.1)

$$S_m = \frac{\lambda}{2}\overline{\varphi}O_m\varphi + \frac{\lambda}{2}\overline{\psi}O_m\psi.$$
(3.2)

Using equation (2.4), and considering the matrix $K = \varphi\otimes\overline{\varphi} - \psi\otimes\overline{\psi}$, we can also write

$$\langle\exp(-S_m)\rangle = \mathcal{N}\exp\left(\frac{\lambda}{2}[(r/2+i\eta)^2 - E^2]\,\text{tr}\,K\right)\int dH\exp(-\text{tr}\,HSH + \lambda E\,\text{tr}\,HK),$$

where $S = \frac{1}{2}\gamma NI + \frac{\lambda}{2}K$. Completing squares, we have

$$\langle\exp(-S_m)\rangle = \mathcal{N}\exp\left(\frac{\lambda}{2}[(r/2+i\eta)^2 - E^2]\,\text{tr}\,K + \frac{\lambda^2}{4}E^2\,\text{tr}\,KS^{-1}K\right)(2/\gamma N)^{\frac{N^2}{2}}I(\tilde{S}),$$
(3.3)

where we have defined

$$I(\tilde{S}) = \int dH\exp(-\text{tr}\,H\tilde{S}H), \qquad \tilde{S} = I + \frac{\lambda}{\gamma N}K.$$
(3.4)

In order to evaluate $I(\tilde{S})$, we will consider the general case where $\tilde{S} = I + g(K)$, substituting $g(x) = \frac{\lambda}{\gamma N}x$ at the end of the calculation.

As the measure $\mathrm{d}H$ is invariant under unitary transformations $H \to UHU^{\dagger}$, $U \in U(N)$, we have, $I(U\tilde{S}U^{\dagger}) = I(\tilde{S})$; and taking the matrix U that diagonalizes $\tilde{S}$: $U\tilde{S}U^{\dagger} = D$, $\tilde{S} = U^{\dagger}DU$ we can write

$$I(\tilde{S}) = I(D) = \int \mathrm{d}H \exp\left(-\sum_i d_i H_{ii}^2 - \sum_{i<j}(d_i + d_j)H_{ij}\overline{H}_{ij}\right)$$

$$= \pi^{\frac{N^2}{2}} \prod_i d_i^{-1/2} \prod_{i<j}(d_i + d_j)^{-1} = \pi^{\frac{N^2}{2}} 2^{\frac{N}{2}} \exp\left(-1/2\sum_{i,j} \ln(d_i + d_j)\right),$$

where $d_i = 1 + g_i$ are the diagonal elements of D, written in terms of the eigenvalues g_i of the matrix $g(K)$, so we obtain

$$I(\tilde{S}) = (\pi/2)^{\frac{N^2}{2}} 2^{\frac{N}{2}} \exp\left(-1/2\sum_{i,j} \zeta\left(\frac{g_i + g_j}{2}\right)\right), \qquad \zeta(x) = \ln(1 + x). \tag{3.5}$$

Now, if we expand $\zeta(x) = \sum_{n=1} a_n x^n$, the exponent in $I(\tilde{S})$ can be written in terms of $U(N)$ invariants,

$$\sum_{i,j} \zeta\left(\frac{g_i + g_j}{2}\right) = \sum_{i,j}\sum_n \frac{a_n}{2^n} \sum_l \binom{n}{l} g_i^l g_j^{n-l}$$

$$= \sum_n \frac{a_n}{2^n} \sum_l \binom{n}{l} \mathrm{tr}(g^l)\,\mathrm{tr}(g^{n-l}). \tag{3.6}$$

In the appendix, we show that the trace of a given function $f(K)$, with $K = \varphi \otimes \overline{\varphi} - \psi \otimes \overline{\psi}$, is given by

$$\mathrm{tr}\, f(K) = Nf_0 + f(\alpha) - f(\beta) + \frac{f'(\alpha) - f'(\beta)}{\alpha - \beta}\mu\overline{\mu},$$

$$\alpha = \varphi \cdot \overline{\varphi}, \qquad \beta = \psi \cdot \overline{\psi}, \qquad \mu = \varphi \cdot \overline{\psi}. \tag{3.7}$$

Using this formula in equation (3.6), we can resume the series in the index n,

$$\sum_{i,j} \zeta\left(\frac{g_i + g_j}{2}\right) = \zeta(g(\alpha)) + 2N\zeta(g(\alpha)/2) + \zeta(g(\beta)) - 2N\zeta(g(\beta)/2) - 2\zeta(g(\alpha)/2$$

$$+ g(\beta)/2) + \frac{\mu\overline{\mu}}{\alpha - \beta}\frac{\mathrm{d}}{\mathrm{d}\alpha}[\zeta(g(\alpha)) + 2N\zeta(g(\alpha)/2) - 2\zeta(g(\alpha)/2 + g(\beta)/2)]$$

$$+ \frac{\mu\overline{\mu}}{\alpha - \beta}\frac{\mathrm{d}}{\mathrm{d}\beta}[\zeta(g(\beta)) - 2N\zeta(g(\beta)/2) - 2\zeta(g(\alpha)/2 + g(\beta)/2)], \tag{3.8}$$

where we have used $g(0) = 0$ and $\zeta(0) = 0$.

Now, from equations (3.5), (3.8), and also considering the formula (3.7) in the first factor of equation (3.3), we finally obtain,

$$\langle\exp(-S_m)\rangle = G_1(\alpha, \beta) + G_2(\alpha, \beta)\mu\overline{\mu}, \tag{3.9}$$

where,

$$G_1 = \frac{1}{2}\exp\left(\frac{\lambda}{2}[(r/2 + i\eta)^2 - E^2](\alpha - \beta) + \frac{\lambda^2 E^2}{2}\left[\frac{\alpha^2}{\gamma N + \lambda\alpha} - \frac{\beta^2}{\gamma N + \lambda\beta}\right]\right)$$

$$\times \frac{(2 + \lambda\beta/\gamma N)^N}{(2 + \lambda\alpha/\gamma N)^N} \frac{(2 + \lambda(\alpha + \beta)/\gamma N)}{(1 + \lambda\alpha/\gamma N)^{1/2}(1 + \lambda\beta/\gamma N)^{1/2}}, \tag{3.10}$$

and

$$G_2 = G_1 \left\{ \frac{\lambda^2 E^2}{2\gamma N} \frac{(2 + \lambda(\alpha + \beta)/\gamma N)}{(1 + \lambda\alpha/\gamma N)^2(1 + \lambda\beta/\gamma N)^2} + \frac{\lambda^2}{\gamma^2 N(2 + \lambda\alpha/\gamma N)(2 + \lambda\beta/\gamma N)} \right.$$
$$\left. + \frac{\lambda^3(\beta - \alpha)}{2\gamma^3 N^3(1 + \lambda\alpha/\gamma N)(1 + \lambda\beta/\gamma N)(2 + \lambda(\alpha + \beta)/\gamma N)} \right\}. \tag{3.11}$$

Note that this expression is correctly normalized as for $\varphi = 0$, $\psi = 0$ we have, $S_m = 0$, $\alpha = 0$, $\beta = 0$ and $\mu = 0$. Then, in this case, we see from equation (3.10) that the first and second members in equation (3.9) are equal to 1.

Now, we can substitute equation (3.9) in the correlator (3.1) to obtain,

$$\langle \text{tr}(O_m^{-1}) \, \text{tr}(O_m^{-1}) + \text{tr}(O_m^{-1} O_m^{-1}) \rangle = (\lambda/2)^2 (-1)^{N(N+1)/2} \pi^{-N}$$
$$\times \int d\varphi \, d\overline{\varphi} \, d\psi \, d\overline{\psi} \alpha^2 [G_1(\alpha, \beta) + G_2(\alpha, \beta)\mu\overline{\mu}]. \tag{3.12}$$

We will still transform this expression to a function of α and β only. In order to do so, we recall that the fermionic integral picks-up the term of the integrand having the form $D\overline{\psi}_1\psi_1\overline{\psi}_2\psi_2\cdots\overline{\psi}_N\psi_N$. Now, in $\beta = \psi \cdot \overline{\psi}$, the ψ_i and $\overline{\psi}_j$ come in pairs $\psi_i\overline{\psi}_i$; then, the crossed terms $(i \neq j)$ in

$$\mu\overline{\mu} = \varphi \cdot \overline{\psi}\overline{\varphi} \cdot \psi = \sum_i \varphi_i\overline{\varphi}_i\overline{\psi}_i\psi_i + \sum_{i \neq j} \varphi_i\overline{\varphi}_j\overline{\psi}_i\psi_j$$

cannot contribute to the fermionic integral, that is

$$\int d\psi \, d\overline{\psi} G_2(\alpha, \beta)\mu\overline{\mu} \int d\psi \, d\overline{\psi} G_2(\alpha, \beta) \sum_i \varphi_i\overline{\varphi}_i\overline{\psi}_i\psi_i$$
$$= \frac{1}{N} \sum_i \varphi_i\overline{\varphi}_i \int d\psi \, d\overline{\psi} \, G_2(\alpha, \beta)\overline{\psi} \cdot \psi = \frac{1}{N}\varphi \cdot \overline{\varphi} \int d\psi \, d\overline{\psi} \, G_2(\alpha, \beta)\overline{\psi} \cdot \psi, \tag{3.13}$$

where we have also used the symmetry among the ψ_i. Putting this information together, we arrive at

$$\langle \text{tr}(O_m^{-1}) \text{tr}(O_m^{-1}) + \text{tr}(O_m^{-1} O_m^{-1}) \rangle = (\lambda/2)^2 (-1)^{N(N+1)/2} \pi^{-N}$$
$$\times \int d\varphi \, d\overline{\varphi} \, d\psi \, d\overline{\psi} \alpha^2 [G_1(\alpha, \beta) - \frac{\alpha\beta}{N} G_2(\alpha, \beta)]. \tag{3.14}$$

Now, the integrals over φ, $\overline{\varphi}$ can easily be expressed in terms of a single-variable integral over α. Using the solid angle in M dimensions, $\Omega_M = 2\pi^{M/2}/\Gamma(M/2)$, and the fermionic integral,

$$\int d\psi \, d\overline{\psi} F(\beta) = (-1)^{N(N+1)/2} \frac{d^N}{d\beta^N} F \bigg|_{\beta=0},$$

we finally obtain

$$\langle \text{tr}(O_m^{-1}) \, \text{tr}(O_m^{-1}) + \text{tr}(O_m^{-1} O_m^{-1}) \rangle$$
$$= \frac{(\lambda/2)^2}{(N-1)!} \int_0^\infty d\alpha \, \alpha^{N+1} \frac{d^N}{d\beta^N} \left[G_1(\alpha, \beta) - \frac{\alpha\beta}{N} G_2(\alpha, \beta) \right]_{\beta=0}. \tag{3.15}$$

4. Analytical determination for the correlators of O_m operators

Here, we will study the analytical properties of the four-point O_m correlators. Recalling that η is a small parameter (we will take the limit $\eta \to 0$), we can drop η^2 in the exponent $\frac{\lambda}{2}(r/2+i\eta)^2\alpha$, in equations (3.10) and (3.11). We will suppose $r = E_1 - E_2 > 0$; then, for the integral representation (3.15) to be well defined, we must take $\lambda = i$. Precisely, this choice produces an associated factor in the bosonic integral,

$$\exp i\omega\alpha \exp(-\eta r\alpha/2), \qquad \omega = \frac{r^2}{8}, \tag{4.1}$$

which leads to a convergent integral.

We underline here that, as usual, the factor N in the exponent of the probability measure in equation (2.8) assures that the energy band attain a finite width when large N matrices are considered. Therefore, the natural scale for the level spacing is $1/N$. For this reason, we will define the variables,

$$r = \frac{s}{N}, \qquad \gamma N\omega = \frac{\chi}{N}, \qquad \chi = \gamma\frac{s^2}{8}. \tag{4.2}$$

Using equations (3.10), (3.11) and (3.15), working around $E = (E_1 + E_2)/2 = 0$, and considering the change of variables $\alpha \to \alpha/\omega$, $\beta \to -i\beta/\omega$, the small η ($\eta \ll r$) contribution of the last term in equation (2.10) to the mixed retarded and advanced correlator results,

$$C(\chi) = -\frac{r^2}{2}\langle \operatorname{tr}(O_m^{-1})\operatorname{tr}(O_m^{-1}) + \operatorname{tr}(O_m^{-1}O_m^{-1})\rangle$$

$$= \gamma N^2 i^N \int_0^\infty d\alpha\, \alpha^{N+1} \exp(i\alpha)\, \frac{1}{N!}\frac{d^N}{d\beta^N}\bigg|_{\beta=0} \exp(-\beta)$$

$$\times \left[\frac{\left(\frac{\chi}{N}+b\right)^{N-1}}{\left(\frac{\chi}{N}+a\right)^{N+1}}\frac{(a+b)^2}{(ab)^{1/2}} - \frac{1}{4\chi}\frac{\left(\frac{\chi}{N}+b\right)^N}{\left(\frac{\chi}{N}+a\right)^N}\frac{(a-\frac{\chi}{N})(b-\frac{\chi}{N})(b-a)}{(ab)^{3/2}}\right], \tag{4.3}$$

where $a = \frac{\chi}{N}+i\alpha$, $b = \frac{\chi}{N}+\beta$.

We see that this expression is a sum of terms containing products of the form $I_\nu^{(n)}Q_\mu^{(n)}$, with $n = 0, 1$ and μ, ν semi-integers, where

$$I_\nu^{(n)}(\chi) = \gamma i^N \int_0^\infty d\alpha\, \frac{\alpha^{N+1}\left(\frac{\chi}{N}+i\alpha\right)^\nu \exp(i\alpha)}{\left(\frac{2\chi}{N}+i\alpha\right)^{N+1-n}}, \tag{4.4}$$

and the Q are polynomials,

$$Q_\mu^{(n)}(\chi) = \frac{1}{N!}\frac{d^N}{d\beta^N}\bigg|_{\beta=0}\left(\frac{\chi}{N}+b\right)^{N-1+n} b^\mu \exp(-\beta). \tag{4.5}$$

Now, the integrand in equation (4.4) presents singularities on the upper complex α-plane, on the imaginary axis. There is an isolated singularity when $\alpha = 2i\chi/N$, and there is a continuum of singularities for $\alpha = iy$, $y > \chi/N$. As no singularities are found on the first quadrant, and the α integral on the arc at infinity goes to zero, we can rotate the integration path until the positive imaginary axis is approached anti-clockwise. Recalling that, from equation (4.1), ω contains a small positive imaginary part that regularizes the integral, according to equation (4.2) we have also to consider, $\chi \to \chi + i\epsilon$. Then, we can use the equivalent representation,

$$I_\nu^{(n)}(\chi) = \gamma(-1)^{N+1}\int_0^\infty dy\, \frac{y^{N+1}\left(\frac{\chi}{N}-y+i\epsilon\right)^\nu \exp(-y)}{\left(\frac{2\chi}{N}-y+i\epsilon\right)^{N+1-n}}. \tag{4.6}$$

We can split this integral into two parts,

$$I_\nu^{(n)} = J_\nu^{(n)} + K_\nu^{(n)}, \tag{4.7}$$

where the integral in the J runs from 0 to χ/N, while the integral in the K runs from χ/N to ∞. A straightforward calculation leads to

$$J_\nu^{(n)}(\chi) = \gamma(-1)^{N+1}(\chi/N)^{n+\nu+1} \int_0^1 dv \frac{v^{N+1}(1-v)^\nu \exp\left(-\frac{\chi}{N}v\right)}{(2-v)^{N+1-n}}.$$

On the other hand, we can write

$$K_\nu^{(n)}(\chi) = \gamma(-1)^{N+1} e^{i\pi\nu} \int_{\frac{\chi}{N}}^\infty dy\, y^{N+1}\left(y - \frac{\chi}{N}\right)^\nu \exp(-y) \frac{1}{(N-n)!} \frac{d^{N-n}}{dy^{N-n}}\left(\frac{1}{\frac{2\chi}{N} - y + i\epsilon}\right).$$

Note that the factor $e^{i\pi\nu}$ is pure imaginary, as the ν are semi-integer. Then, using $(2\chi/N - y + i\epsilon)^{-1} = P(2\chi/N - y)^{-1} - i\pi\delta(2\chi/N - y)$, we see that the real part of the K can be easily computed, as this is associated with the δ term,

$$\mathrm{Re}\, K_\nu^{(n)}(\chi) = e^{i\pi(\nu-n+\frac{1}{2})} \gamma\pi \frac{1}{(N-n)!} \frac{d^{N-n}}{dy^{N-n}}\bigg|_{y=\frac{2\chi}{N}} y^{N+1}\left(y - \frac{\chi}{N}\right)^\nu \exp(-y), \tag{4.8}$$

and changing variables, $y = 2\chi/N + \beta$,

$$\mathrm{Re}\, K_\nu^{(n)}(\chi) = e^{i\pi(\nu-n+\frac{1}{2})} P_\nu^{(n)}(\chi), \tag{4.9}$$

where we have introduced the functions,

$$P_\nu^{(n)}(\chi) = \gamma\pi \exp(-2\chi/N) \frac{1}{(N-n)!} \frac{d^{N-n}}{d\beta^{N-n}}\bigg|_{\beta=0} \left(\frac{\chi}{N} + b\right)^{N+1} b^\nu \exp(-\beta), \tag{4.10}$$

and b has been defined after equation (4.3).

5. Correlations for level spacing

In this section, in order to test our single superfield method, we will first obtain an approximate relationship between the O_m-correlators and the DOS–DOS correlators, which becomes exact in the $N \to \infty$ limit. Then, we will compare the finite N O_m-correlators, as N becomes large, with the well-known universal results for DOS–DOS correlations.

The density of states operator, at energy E_1, is given by

$$\hat{\rho}(E_1) = \frac{1}{N} \sum_i \delta(E_1 - E_i) = \frac{1}{2\pi i N} \mathrm{tr}\left(O_{1+}^{-1} - O_{1-}^{-1}\right), \tag{5.1}$$

(cf equation (2.2)). Then, the DOS–DOS correlator can be written as

$$\rho(E_1, E_2) = \langle\hat{\rho}(E_1)\hat{\rho}(E_2)\rangle - \langle\hat{\rho}(E_1)\rangle\langle\hat{\rho}(E_2)\rangle$$

$$= \frac{1}{2\pi^2 N^2} \mathrm{Re}\left(\langle \mathrm{tr}\left(O_{1+}^{-1}\right)\mathrm{tr}\left(O_{2-}^{-1}\right)\rangle - \langle\mathrm{tr}\left(O_{1+}^{-1}\right)\rangle\langle\mathrm{tr}\left(O_{2-}^{-1}\right)\rangle\right)$$

$$- \frac{1}{2\pi^2 N^2} \mathrm{Re}\left(\langle\mathrm{tr}\left(O_{1+}^{-1}\right)\mathrm{tr}\left(O_{2+}^{-1}\right)\rangle - \langle\mathrm{tr}O_{1+}^{-1}\rangle\langle\mathrm{tr}O_{2+}^{-1}\rangle\right). \tag{5.2}$$

As we are interested in analysing the new single superfield representation for mixed correlators, we will simplify the standard part of the calculation as much as possible. In particular, in the $N \to \infty$ limit, the following factorization of retarded–retarded correlators is verified (see for example [4, 17]),

$$\langle\mathrm{tr}\left(O_{1+}^{-1}\right)\mathrm{tr}\left(O_{2+}^{-1}\right)\rangle = \langle\mathrm{tr}\, O_{1+}^{-1}\rangle\langle\mathrm{tr}\, O_{2+}^{-1}\rangle. \tag{5.3}$$

In this manner, up to irrelevant terms that scale to zero for large values of N, we can consider the approximation,

$$\rho(E_1, E_2) \approx \frac{1}{2\pi^2 N^2} \text{Re}\left(-r^2/2\langle \text{tr}(O_m^{-1})\text{tr}(O_m^{-1}) + \text{tr}(O_m^{-1}O_m^{-1})\rangle\right) + \frac{R}{2\pi^2 N^2}, \tag{5.4}$$

$$\frac{R}{N^2} \approx \frac{1}{N^2} \text{Re}\left(\frac{1}{2}\langle \text{tr}(O_{1+}^{-1})\rangle^2 + \frac{1}{2}\langle \text{tr}(O_{2-}^{-1})\rangle^2 - \langle \text{tr}(O_{1+}^{-1})\rangle\langle \text{tr}(O_{2-}^{-1})\rangle\right)$$
$$+ \frac{1}{N^2}\text{Re}\left(\frac{1}{2}\langle \text{tr}(O_{1+}^{-1}O_{1+}^{-1})\rangle + \frac{1}{2}\langle \text{tr}(O_{2-}^{-1}O_{2-}^{-1})\rangle - \langle \text{tr}(O_{1+}^{-1}O_{2-}^{-1})\rangle\right), \tag{5.5}$$

where we have used equations (2.10), (5.3), and $r \neq 0, \eta \to 0$.

In addition, we have

$$\langle \text{tr}(O_{1+}^{-1}O_{1+}^{-1})\rangle = \left\langle \sum_i (E_i - E_1 - i\eta)^{-2} \right\rangle = \frac{\mathrm{d}}{\mathrm{d}E_1}\langle \text{tr}(O_{1+}^{-1})\rangle,$$

$$\langle \text{tr}(O_{2-}^{-1}O_{2-}^{-1})\rangle = \frac{\mathrm{d}}{\mathrm{d}E_2}\langle \text{tr}(O_{2-}^{-1})\rangle, \tag{5.6}$$

while, using equation (2.9) for $r \neq 0$, the last term in equation (5.5) can be separated in only retarded and only advanced parts,

$$\langle \text{tr}(O_{1+}^{-1}O_{2-}^{-1})\rangle = \frac{1}{r}\langle \text{tr}(O_{1+}^{-1})\rangle - \frac{1}{r}\langle \text{tr}(O_{2-}^{-1})\rangle. \tag{5.7}$$

Note that, using (5.2), (2.10), (5.6) and (5.7), the density–density correlator can be generally written in terms of four-point O_m correlators, up to only retarded and only advanced terms which can be studied by following standard procedures.

In particular, in the random matrix case, we quote the well-known large N results,

$$\frac{1}{N}\text{tr}(O_{1+}^{-1}) = -\frac{\gamma}{2}E_1 + i\pi\rho(E_1), \qquad \frac{1}{N}\text{tr}(O_{2-}^{-1}) = -\frac{\gamma}{2}E_2 - i\pi\rho(E_2),$$

$$\rho(E) = \frac{\gamma}{2\pi}\sqrt{\frac{4}{\gamma} - E^2}, \qquad\qquad E^2 < \frac{4}{\gamma}. \tag{5.8}$$

Then, using equations (5.6) and (5.7), we see that the three terms in the second line of equation (5.5), when multiplied by the $1/N^2$ factor, are irrelevant. In fact, the second line tends to zero faster than $1/N$, as from equations (5.6) and (5.7),

$$\frac{1}{N}\text{Re}\langle \text{tr}(O_{1+}^{-1}O_{1+}^{-1})\rangle \approx -\frac{\gamma}{2}, \qquad \frac{1}{N}\text{Re}\langle \text{tr}(O_{2-}^{-1}O_{2-}^{-1})\rangle \approx -\frac{\gamma}{2},$$

$$\frac{1}{N}\text{Re}\langle \text{tr}(O_{1+}^{-1}O_{2-}^{-1})\rangle \approx -\frac{\gamma}{2r}(E_1 - E_2) = -\frac{\gamma}{2},$$

thus, implying that a $1/N$ factor multiplying the sum in the parenthesis of this line would already give a vanishing large-N result.

With regard to the first line in equation (5.5), let us recall that, as explained after equation (4.1), the appropriate scaling to study level spacing is $r = s/N$, with s fixed. This is the natural variable we have used in the mixed correlator in equation (4.3) ($\chi = \gamma s^2/8$), which we are studying at $E_1 + E_2 = 0$, that is, $E_1 = s/(2N)$, $E_2 = -s/(2N)$. As a consequence, in the first line of equation (5.5), we can use, $(1/N)\langle \text{tr}(O_{1+}^{-1})\rangle \approx i\pi\rho(0)$ and $(1/N)\langle \text{tr}(O_{2-}^{-1})\rangle \approx -i\pi\rho(0)$, that is,

$$\frac{R}{2\pi^2 N^2} \approx \frac{1}{2\pi^2}\left(\frac{1}{2}(i\pi\rho(0))^2 + \frac{1}{2}(-i\pi\rho(0))^2 - (i\pi\rho(0))(-i\pi\rho(0))\right)$$

$$\approx -(\rho(0))^2. \tag{5.9}$$

Then, using this information in equation (5.4), we find

$$1 + \frac{\rho(E_1, E_2)}{\rho(E_1)\rho(E_2)} \approx 1 + \frac{\rho(E_1, E_2)}{(\rho(0))^2}$$

$$\approx \frac{1}{2\pi^2(\rho(0))^2 N^2} \operatorname{Re}\left(-r^2/2\langle \operatorname{tr}(O_m^{-1})\operatorname{tr}(O_m^{-1}) + \operatorname{tr}(O_m^{-1}O_m^{-1})\rangle\right),$$

$$\approx \frac{1}{2\gamma N^2} \operatorname{Re}(C(\chi)), \tag{5.10}$$

where we have used $\rho(0) = \sqrt{\gamma}/\pi$ (cf equation (5.8)) and equation (4.3).

In this manner, we see that the large-N four-point correlator of O_m operators in equation (5.10) gives the usual $\langle \hat{\rho}(E_1)\hat{\rho}(E_2)\rangle/(\rho(E_1)\rho(E_2))$ correlator.

In order to test our single superfield representation for mixed correlators, we can compare the large-N behaviour for the last line in equation (5.10) and the well-known ($N \to \infty$) universal behaviour for the left member in that equation,

$$1 + \frac{\rho(E_1, E_2)}{\rho(E_1)\rho(E_2)} \approx 1 - \left(\frac{\sin\sqrt{\gamma}s}{\sqrt{\gamma}s}\right)^2 \tag{5.11}$$

(note that $\sqrt{\gamma}s = \pi r/\Delta$, where $\Delta = 1/(\rho(0)N)$ is the mean level spacing).

As we have seen in section 4, $C(\chi)$ is a sum of terms containing products of the form $I_\nu^{(n)}Q_\mu^{(n)}$, with real coefficients. The polynomials $Q_\mu^{(n)}$ are real (cf equation (4.5)), so that $\operatorname{Re} C(\chi)$ depends on the real part of the I, which have been split into two terms in equation (4.7), by using the $J_\nu^{(n)}$ and $K_\nu^{(n)}$ functions. It can be seen that the J contribution to $\operatorname{Re} C(\chi)$ is suppressed for large N. On the other hand, the real part of the K has been evaluated in closed form in equation (4.9). Putting all this information together, we arrive at

$$\frac{1}{2\gamma N^2} \operatorname{Re}(C(\chi)) \approx \frac{1}{2\gamma} \left(P_{3/2}^{(0)}Q_{-1/2}^{(0)} - 2P_{1/2}^{(0)}Q_{1/2}^{(0)} + P_{-1/2}^{(0)}Q_{3/2}^{(0)} + \frac{1}{4\chi}P_{-1/2}^{(1)}Q_{1/2}^{(1)} \right.$$

$$+ \frac{1}{4\chi}P_{1/2}^{(1)}Q_{-1/2}^{(1)} + \frac{1}{4N}P_{-3/2}^{(1)}Q_{1/2}^{(1)} - \frac{1}{4N}P_{1/2}^{(1)}Q_{-3/2}^{(1)}$$

$$\left. - \frac{\chi}{4N^2}P_{-3/2}^{(1)}Q_{-1/2}^{(1)} - \frac{\chi}{4N^2}P_{-1/2}^{(1)}Q_{-3/2}^{(1)} \right). \tag{5.12}$$

In figure 1, we display, in the range $s \in [0, 12]$, the $N \to \infty$ universal law in equation (5.11) and our $N = 10$ and $N = 70$ correlator $(2\gamma N^2)^{-1}\operatorname{Re}(C(\chi))$ in equation (5.12). We see that the correlators approach the correct asymptotic form as N increases for all values of the scaled variable s. For small values of s the asymptotic law is well approximated even with small values of N. As N increases, larger values of s progressively fit the $N \to \infty$ behaviour.

To show this effect in more detail we present in figure 2 a set of figures in the range $s \in [2, 12]$ where we display a detailed view of the universal law and the convergence of our finite-N correlator, by considering $N = 10, 30, 50$ and 70. We can see that for larger values of N more oscillations are accommodated, clearly showing the onset of the asymptotic universal law. We also note in this figure, that in a region close to the origin the convergence is very fast. The parameter γ, that characterizes the GUE, has been taken equal to 1. However, the same matching has also been verified when γ is changed. In this set of figures, we have also displayed the corresponding finite N DOS–DOS correlators obtained in terms of orthogonal polynomials, as given in Mehta's book [18]. Of course, the fitting of these polynomials with the universal law is also improved as N increases. Although both curves converge to the asymptotic universal law, the finite N results for $(2\gamma N^2)^{-1}\operatorname{Re}(C(\chi))$ do not coincide with the finite-N exact DOS–DOS correlators. This is expected; as we have seen in section 5, both quantities differ by terms which only become zero in the $N \to \infty$ limit.
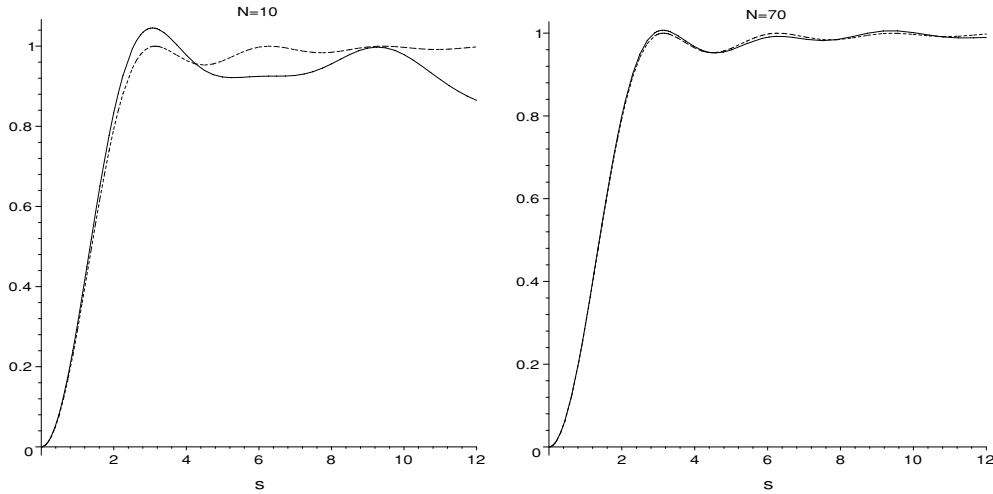
Figure 1. In these two figures the bold line represents the correlator (5.12), for $N = 10$ and $N = 70$ respectively. The dashed line corresponds to the $1 - (\sin^2 s)/s^2$ law in equation (5.11). We have considered $\gamma = 1$ in the two figures.

6. Boson–fermion symmetry and four-point O_m correlators in QH systems

In this section, we will consider a planar system of electrons in the presence of a magnetic field B and impurities represented by a random potential $V(\mathbf{x})$. The corresponding Hamiltonian operator is

$$H = H_0 + V, \qquad H_0 = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2, \tag{6.1}$$

where A is the vector potential (we will use the symmetric gauge).

Here, we can also introduce the quadratic operator O_m according to equation (2.4). As discussed, because of Wick's theorem, the four-point correlators associated with O_m will encode important information about the mixed retarded and advanced products appearing in the density of localized states in equation (1.2). In a similar way to equation (2.12), we can introduce a single superfield with components $\varphi(\mathbf{x})$, $\psi(\mathbf{x})$ whose action is given by

$$S_m = \frac{\lambda}{2} \int \mathrm{d}^2\mathbf{x} \, (\overline{\varphi} O_m \varphi + \overline{\psi} O_m \psi), \tag{6.2}$$

thus leading to a properly normalized representation of the four-point correlators containing G_m Green's functions.

For a very strong magnetic field, the transitions between Landau levels are suppressed. Then, if we are interested in the case where the first Landau level is filled, we can consider the projection,

$$\varphi = \mathrm{e}^{-\frac{1}{4}\kappa^2 z\bar{z}} u(z), \qquad \psi = \mathrm{e}^{-\frac{1}{4}\kappa^2 z\bar{z}} v(z), \tag{6.3}$$

where $z = x + \mathrm{i}y$ and $\kappa = eB/\hbar$.

The action S_m for the projected fields can be written as

$$S_m = \frac{\lambda}{2} \int \mathrm{d}z \, \mathrm{d}\bar{z} \left[\left(V + \frac{\hbar eB}{4m} - E \right)^2 - (r/2 + \mathrm{i}\eta)^2 \right] \alpha(z, \bar{z}),$$

$$\alpha(z, \bar{z}) = \mathrm{e}^{-\frac{1}{2}\kappa^2 z\bar{z}} (\overline{u}u + \overline{v}v). \tag{6.4}$$
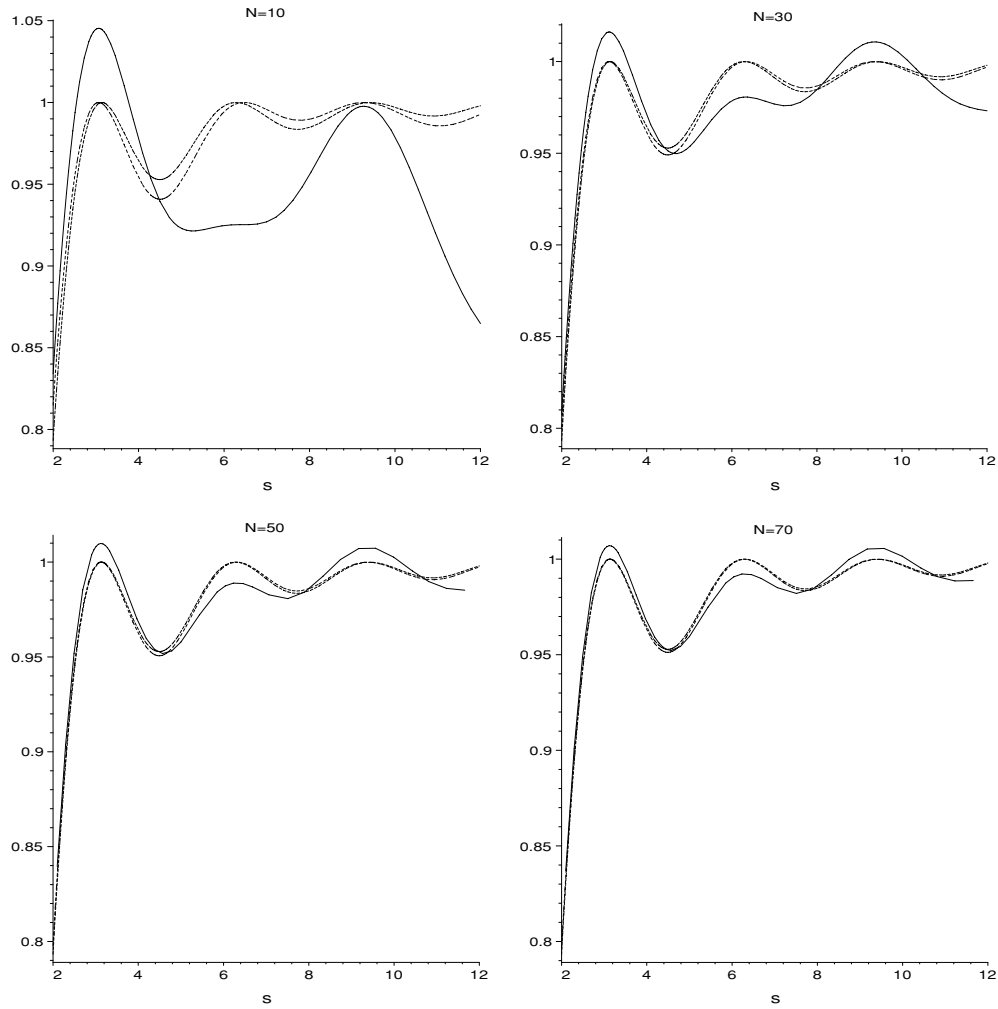
Figure 2. In this set of figures the bold line represents the correlator (5.12), for $N = 10, 30, 50$ and 70. The dashed line corresponds to the $1 - (\sin^2 s)/s^2$ law in equation (5.11), while the dotted line corresponds to the N DOS–DOS correlator obtained in terms of orthogonal polynomials [18]. We have considered $\gamma = 1$ in the four figures.

We would like to underline that equation (6.4) is local in the random potential (containing up to quadratic terms). Then, if the potentials are uncorrelated at different points, the corresponding average will lead to an effective action which is local in the field $\alpha(z, \bar{z})$,

$$S_m^{\text{eff}} = \frac{\lambda}{2} \int \mathrm{d}z \, \mathrm{d}\bar{z} \, g(\alpha), \tag{6.5}$$

where $g(\alpha)$ depends on the type of disorder.

Then, we will be able to follow [14, 15], defining a superfield $\Phi(z, \theta) = u(z) + 1/\sqrt{2}\kappa\theta v(z)$ to write

$$\alpha(z, \bar{z}) = \frac{2\pi}{\kappa^2} \int \mathrm{d}\theta \, \mathrm{d}\bar{\theta} \, \mathrm{e}^{-\frac{1}{2}\kappa^2(z\bar{z}+\theta\bar{\theta})} \Phi\bar{\Phi}, \tag{6.6}$$

(the normalization for the Grassmann measure is $\int d\theta \, d\bar{\theta} \, e^{-\theta\bar{\theta}} = 1/\pi$) and construct a supersymmetric representation,

$$S_m^{\text{eff}} = \frac{\lambda}{2} \frac{2\pi}{\kappa^2} \int dz \, d\bar{z} \, d\theta \, d\bar{\theta} \, e^{-\frac{1}{2}\kappa^2(z\bar{z}+\theta\bar{\theta})} h(\Phi\bar{\Phi}), \qquad h(\alpha) = \int_0^\alpha \frac{d\beta}{\beta} g(\beta), \qquad (6.7)$$

with the remarkable boson–fermion symmetry appearing in [15], when computing the total density of states in QH systems. This is an interesting result as this is the underlying symmetry behind the exact expression for the total density.

As discussed in that paper, if the traditional approach to treat mixed retarded and advanced correlators is followed, the boson–fermion symmetry is lost because of the superfield doubling used to represent each prescription. Here, we have seen that this important symmetry is restored if the single superfield representation for mixed correlators is considered instead.

7. Conclusions and perspectives

In this work we have introduced a new representation for mixed retarded and advanced four-point correlators. These quantities are the relevant ones when studying localization and transport properties in disordered systems. On the other hand, these are precisely the kind of correlators which are more difficult to deal with. This comes about from the mixed prescription involved, which leads to a doubling of the field copies or the superfield representation in the traditional approaches.

Here, we have considered a single Green's function G_m for an operator O_m, quadratic in the system's Hamiltonian, which already contains the above-mentioned mixing. In this manner, we have been able to encode the nontrivial information in four-point correlators essentially using a single prescription. In general, this O_m correlator differs from the mixed four-point correlator or the density–density correlator by only retarded and only advanced terms, which can be computed by standard procedures.

These ideas have been tested in the simpler context of random matrix theory. To this aim, we have obtained in section 5, in the $N \to \infty$ limit, a direct relationship between the density–density correlator,

$$\langle \hat{\rho}(E_1)\hat{\rho}(E_2)\rangle/(\rho(E_1)\rho(E_2)),$$

and our four-point correlator for O_m operators, whose GUE average has been computed in section 3. Our closed expression (5.12) for the (finite N) O_m correlator clearly shows the onset of the well-known asymptotic level spacing correlation.

Then, we have seen that the single superfield method is an interesting alternative to be considered when studying disorder. In particular, in the context of quantum Hall systems, we have shown in section 6 that the boson–fermion symmetry is restored for correlators encoding information about mixed retarded and advanced Green's functions, thus opening the possibility of improving our understanding of localization [19].

Appendix

Here we show how to compute traces of matrices which are functions of $K = \varphi \otimes \overline{\varphi} - \psi \otimes \overline{\psi}$.
For a function $f(x)$, with a corresponding expansion $f(x) = \sum_n f_n x^n$, we have, $\operatorname{tr} f(K) = \sum_n f_n \operatorname{tr} K^n$. On the other hand, a direct computation shows that

$$\operatorname{tr} K = \alpha - \beta, \qquad \operatorname{tr} K^2 = \alpha^2 + 2\mu\overline{\mu} - \beta^2, \qquad \operatorname{tr} K^3 = \alpha^3 + 3(\alpha + \beta)\mu\overline{\mu} - \beta^3, \quad \text{(A.1)}$$

where,

$$\alpha = \varphi \cdot \overline{\varphi}, \quad \beta = \psi \cdot \overline{\psi}, \quad \mu = \varphi \cdot \overline{\psi}. \tag{A.2}$$

It is easy to see the general form of the trace for a given power of K,

$$\operatorname{tr} K^n = \alpha^n + n(\alpha^{n-2} + \alpha^{n-3}\beta + \alpha^{n-4}\beta^2 + \cdots + \alpha^2\beta^{n-4} + \alpha\beta^{n-3} + \beta^{n-2})\mu\overline{\mu} - \beta^n,$$

or in an equivalent form

$$\operatorname{tr} K^n = \alpha^n + n\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\mu\overline{\mu} - \beta^n, \qquad n \neq 0. \tag{A.3}$$

Then, we arrive at a formula that gives the trace of a function of K just in terms of the $U(N)$-invariants (A.2):

$$\begin{aligned}
\operatorname{tr} f(K) &= \operatorname{tr}\left(f_0 I + \sum_{n=1}^{\infty} f_n K^n \right) \\
&= N f_0 + \sum_{n=1}^{\infty} f_n \left(\alpha^n + n\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\mu\overline{\mu} - \beta^n \right) \\
&= N f_0 + f(\alpha) - f(\beta) + \frac{f'(\alpha) - f'(\beta)}{\alpha - \beta}\mu\overline{\mu}.
\end{aligned} \tag{A.4}$$

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